on an automatic detector. In most applications this is more easily said than done and often we have to work from the time-series alone. This would be fine if vehicles don't pass because then (barring detector errors) as soon as one of the trajectories is identified the rest follow. It would also be fine if the observation stations were to be so closely spaced that only small vehicular speed changes over the detector spacing could arise because then it would also be obvious which vehicles are which.

Another form of data (used in connection with freeway studies) arises from time-lapse aerial photographs. Because each photograph is taken at a specific time, t, it is associated with a 'vertical' line on the time-space diagram, as shown in Fig. 1.3b. One can then display by means of dots on the line the location of the 'noses' (or 'tails') of every vehicle at each sampled instant. The photographs automatically display vehicle 'signatures', thanks to their pictorial detail, and this makes it possible (although very tedious and impractical) to connect the appropriate points with smooth lines to develop the vehicle trajectories. This method of construction, however, illustrates that the time-space diagram is a complete summary of the 1-dimensional progress of our vehicles. We note that Fig. 1.3b could have been obtained by actually laying the strips of film side by side and that if these were viewed across a vertical slit that was moved from left to right at an appropriate rate, one would be replaying a movie of the system's evolution! In other words the (t, x) diagram gives a complete description of the history of our vehicles' longitudinal motion.

Besides displaying field data in a complete way, the recipes for constructing Fig. 1.3a and b are also important because they indicate a reverse way in which the (t, x) diagram can be 'read'. In particular note that a horizontal line through the diagram (e.g. at position x_3 in Fig. 1.3a) identifies the times at which successive vehicles pass a stationary observer, and that a vertical line at a given abscissa (e.g. time t_4 in Fig. 1.3b) identifies the vehicle positions at the given time. The truth of this statement does not depend on how the (t, x) diagram was developed. The times between consecutive vehicle observations at a fixed location, h_i, are usually called *headways*, and the distance separations between consecutive vehicles at a given instant, s_j, *spacings*.

1.2.2 Definitions of traffic stream features

The number of vehicles observed by a stationary observer during a given time interval, m, divided by the length of the time interval, T, is the

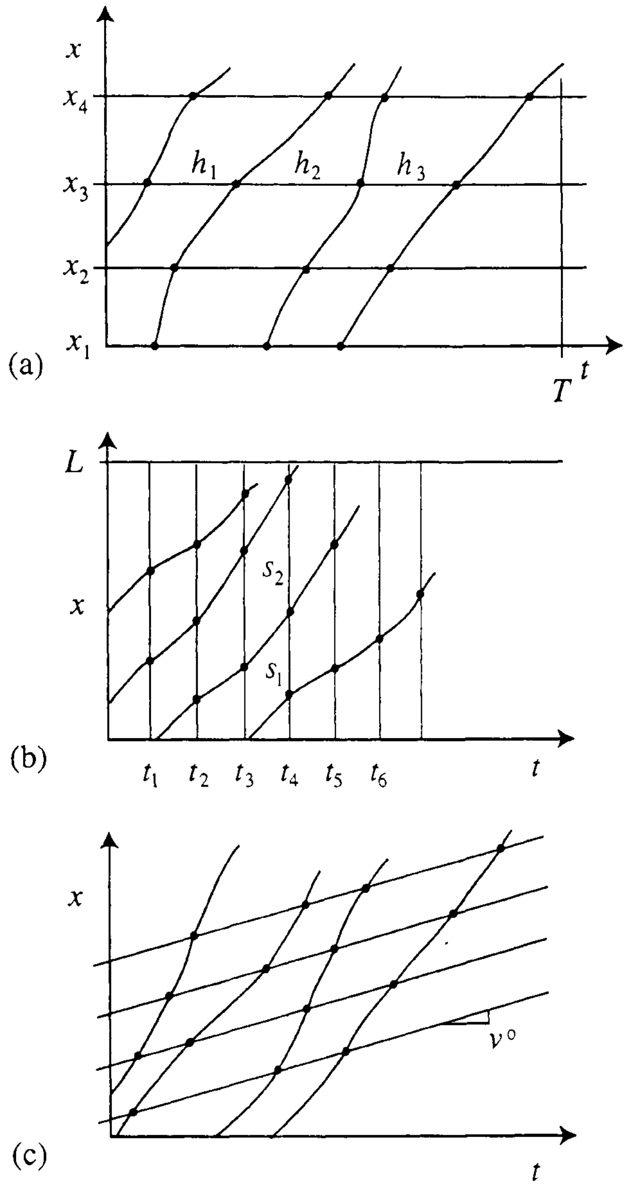


Figure 1.3 Three ways of gathering (t, x) trajectory data: (a) roadside observers at various locations; (b) aerial photographs at different instants; (c) moving observers.

flow, $q = m/T$, for the interval; e.g. the observer at $x = x_3$ in Fig. 1.3a observes a flow, $q = 4/T$, during $0 \leq t \leq T$. It should be clear from Fig. 1.3a that for long observation periods including many vehicles (m, $T \to \infty$) with comparable headways,

$$T \approx \sum_{i=1}^{m} h_i,$$

and therefore, on dividing both sides of this expression by m, we obtain the important relation:

$$q^{-1} = \frac{T}{m} \approx \frac{1}{m} \sum_{i=1}^{m} h_i = \bar{h}\,; \tag{1.10}$$

i.e., under the conditions stated, the flow over an interval is approximately equal to the reciprocal of the average headway seen by a stationary observer during the interval. We note that this relationship is exact if the observation period starts and ends immediately before the arrival of a vehicle. The concept of flow is equivalent to the terms '*volume*', used in certain traffic engineering circles, and '*frequency*', used in connection with scheduled transportation modes.

A similar treatment of the number of vehicles seen on a photograph, n, over a stretch of road of a given length, L, leads to the concept of *density*, $k = n/L$, over the stretch and a parallel relationship of the density with the average spacing:

$$k^{-1} = \frac{L}{n} \approx \frac{1}{n} \sum_{j=1}^{n} s_j = \bar{s}. \tag{1.11}$$

As with headways, the quality of the approximation improves for $L \to \infty$, and the relationship becomes exact when both ends of the interval are immediately ahead of a vehicle.

It should be noted here that other vehicle characteristics (besides spacings and headways) can be averaged across space or time as well; e.g. vehicle occupancies, speeds, etc... and that there is no a-priori reason to expect averages taken across space or time to be the same. Averages taken at a specific location (with time-varying over an interval) are called 'time-means', whereas those taken at an instant over a space interval are termed 'space-means'; e.g. space-mean speed, $\bar{v}_s$, and time-mean speed, $\bar{v}_t$, are the terms used to denote the speed averages obtained in the aforementioned way.

Fig. 1.3c describes one more way in which trajectory data can be recorded (and in which the t, x diagram can be interpreted). It involves observers traveling at a constant speed v° that record the times at which vehicles pass them. The observer trajectories are then plotted and used to locate points in the (t, x) plane through which the vehicle trajectories must pass. In Fig. 1.3c traffic passes the slow-moving observer, but similar figures could be drawn for observers moving faster than traffic and moving against traffic; e.g. if observers on a fast-moving car pool (or contra-flow) lane record the times at which they pass individual vehicles on the general use lanes. Note that the interpretation of Fig. 1.3c generalizes the prior two interpretations because $v^\circ = 0$ leads to Fig. 1.3a and $v^\circ \to \infty$ to Fig. 1.3b.

1.3 Applications of the (t, x) diagram

Here we present two applications of the time-space diagram. The first application is a preview of traffic flow theory for an idealized case that, despite its simplicity, clearly reveals some interesting relationships between traffic flow variables; in this application, the (t, x) diagram helps in the mathematical development, but most importantly it shows physically why the derived expressions are true. The second application is a scheduling problem where vehicles compete for a common right-of-way; there the (t, x) diagram is also used as an aid for thinking that helps eliminate mistakes, and just as importantly, it can be used as an elegant way of displaying the solution that could be used in a professional report.

1.3.1 Traffic flow theory with straight trajectories

We consider a section of road length L that is observed for time T and assume that vehicles travel over the section (approximately) at constant speed without interacting with one another. This scenario could arise in lightly traveled multi-lane freeways with fast and slow vehicles, and in airport corridors with mechanical transportation devices that only a fraction of the people use.

We will also assume that there are only a finite number of speeds v_l that vehicles adopt and that the trajectories of each vehicle family are evenly spaced straight lines. This means that all the vehicles of family 'l' have the same headway *within the family*, h_l. Here, h_l denotes the time separation between two consecutive vehicles of family l; the headway between consecutive vehicles will in general be smaller and will not be constant. This can be seen clearly from the diagram of Fig. 1.4 for the special case where there are two vehicle classes, $l = 1, 2$.

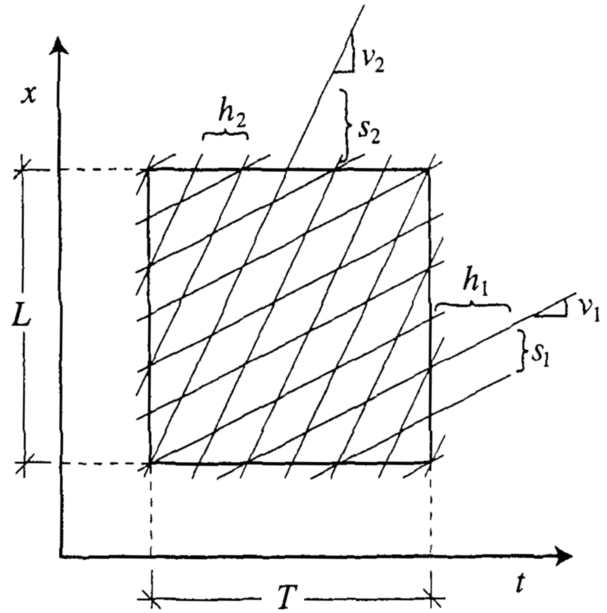


Figure 1.4 Time-space trajectories of two vehicle families.

It can be seen from the geometry of the figure that $h_l v_l = s_l$ for each vehicle class, l, where s_l is the spacing for the class. If the class flows, q_l, and densities, k_l, are defined over intervals containing many vehicles, we can accurately rewrite this relation as:

$$q_l \approx v_l k_l \tag{1.12}$$

by virtue of (1.10) and (1.11).[7] If (1.12) is now added across l, and we recognize that the total flow and density are:

$$q = \sum_l q_l \text{ and } k = \sum_l k_l,$$

we find that:

$$q \approx k \sum_l v_l (k_l / k) = \bar{v}_s k. \tag{1.13}$$

The second equality is justified on noting that the summation in the middle member of (1.13) defines a weighted average of the vehicle speeds where the weighting factors are the fraction of vehicles by type *seen in an aerial photograph.* (This statement follows from the definition of density, given earlier.)

The emphasis is given because the fractions of vehicle types seen in an aerial photograph are usually different from those that would be counted by a stationary observer (q_l/q). To see this intuitively you should refer to Fig. 1.4 and note that a stationary observer sees approximately two fast vehicles for every slow one, but a photograph would show two slow ones for every fast one! You should ponder why, and realize that the stationary observer will invariably see higher fractions of fast vehicles than shown in the aerial photos, independent of the specific speeds, flows and densities of the vehicle families. (Can you imagine what would happen if one of the families had $v_l = 0$; e.g. it corresponded to parked cars?). A related result which should come as no surprise is that the inequality $\bar{v}_t \geq \bar{v}_s$ is generally true[8] for long observation intervals when traffic behaves as described in this section.

A similar disparity should also be expected between time and space averages of other quantities that vary across families, but remain constant within a family. For example, if the fast vehicles of Fig. 1.4 are car-pools (with a 2 person vehicle occupancy) and the slow vehicles are driven without passengers, it should be clear that the average vehicle occupancy will be different depending on the method of observation. Can you figure out what the average vehicle occupancy measured by a moving observer with speed v° (for $v^\circ = 0$, v_1, v_2 and ∞) would be in the example of Fig. 1.4?

The same multiplicity of averages would be obtained for other measures such as energy consumption and pollution generation that vary across vehicle classes with different speeds (e.g. buses and cars; commercial jets and private airplanes, etc...). Which is the 'real average' then? The answer to this question cannot be given absolutely. It depends on the practical problem that motivates your particular analysis and this is why it is important to understand the fundamentals.

1.3.2 Closed loops

It should also be noted that the (t, x) diagram can be used to describe closed loop systems. If we use x to denote the position of a vehicle within the loop ($0 \leq x \leq L$, where L is the length of the loop) then the vehicle's trajectory will 'disappear' upon reaching the coordinate $x = L$, and will simultaneously reappear at $x = 0$. The trajectory of a vehicle that travels at a constant speed along the loop then adopts a 'saw-tooth' shape as shown in Fig. 1.5.

The figure depicts the (parallel) trajectories of 4 vehicles equally spaced on the loop. Such a diagram could represent the behavior of

Outline

1. **Basic assumptions of traffic flow theory**
   a. **Key variables**
   b. Time vs space means
2. Fundamental diagrams (FDs)
3. Highway delay problem

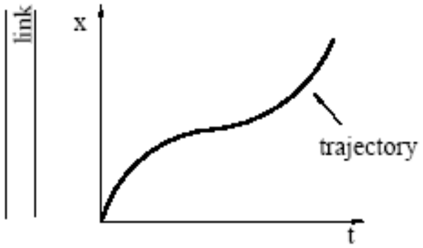
Traffic flow theory

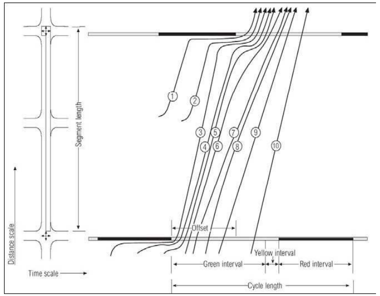
- Today: From traffic flow (traffic streams) to traffic flow theory

- **Traffic flow theory**:
  - Models and hypotheses for explaining traffic flow
  - I.e., what would happen to traffic streams if they were to flow on roads under different conditions, potentially not yet observed

- Models vs data

$$a_{\mathrm{IDM}}(s, v, \Delta v) \longrightarrow$$





Basic assumptions

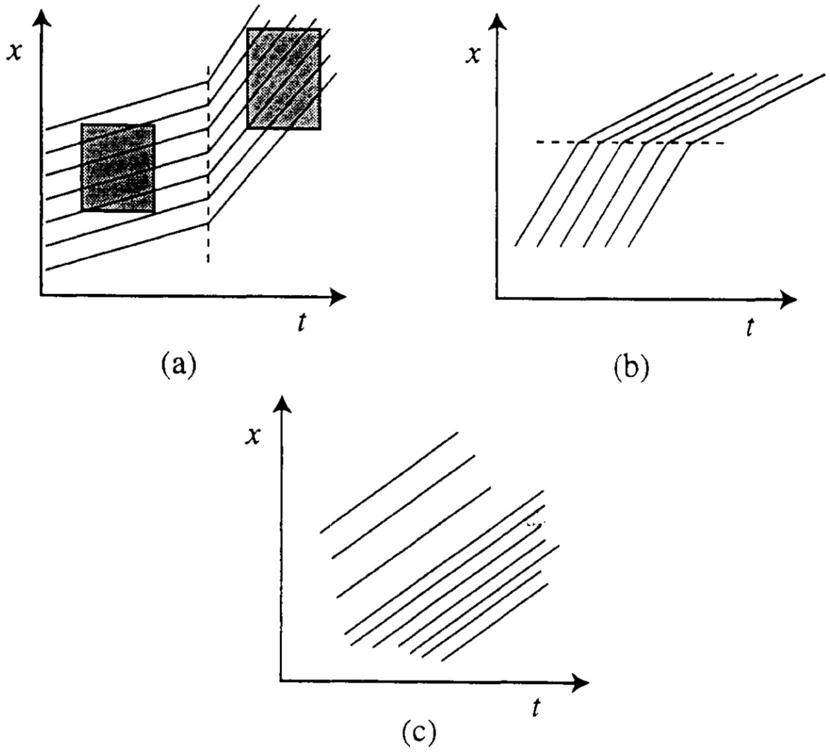
1. Study of a single traffic stream, flowing on a facility with a single entrance and a single exit
2. Uninterrupted traffic
   - Traffic regulated by interactions between vehicles, as opposed to being regulated by external means
   - E.g. on a highway or at unsignalized intersections, as opposed to traffic lights, stop signs.
3. Stationary traffic conditions (vs. time and space-varying dynamics)

Stationary vs non-stationary traffic

Stationary traffic conditions

(vs. time- or space-varying dynamics):

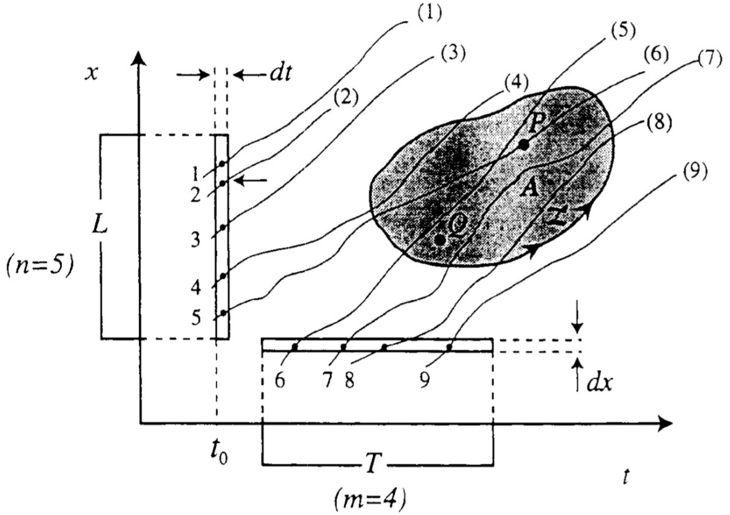
- Traffic is stationary if it is a superposition of families of trajectories that are each parallel and equidistant.



Examples of non-stationary traffic

Traffic stream variables

- Main variables
  - Flow
  - Time headway
  - Density
  - Spacing
  - Speed (space-mean, time-mean)

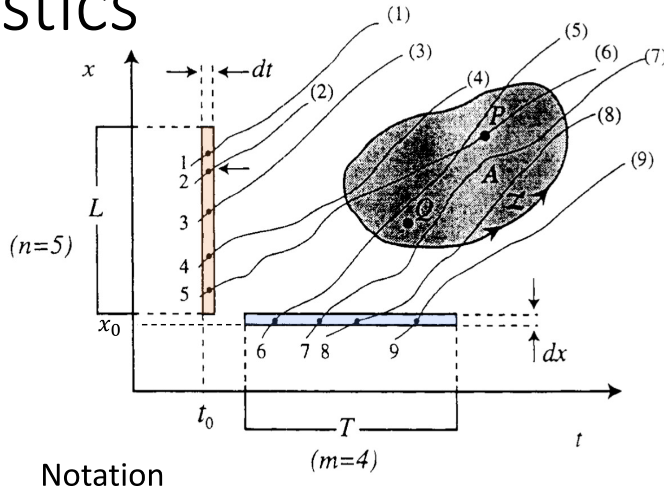


- Aim: Obtain relationships that hold "on average"; i.e. for large stationary time-space regions containing many vehicles

Formulas for traffic characteristics

Table 4.1.. Generalized formulas for various traffic characteristics using two observation methods. Boxed expressions correspond to the original definitions introduced in Chapter 1:

	Method of Observation	
	Instantaneous photograph at time t_0 **(section length, L)**	**Observation from a fixed location** x_0 **(duration, T)**
Density, k(**A**)	$\boxed{n/L}$	$\frac{1}{T}\sum_{j=1}^{m} p_j = \frac{1}{T}\sum_{j=1}^{m}\frac{1}{u_j}$
Flow, q(**A**)	$\frac{1}{L}\sum_{i=1}^{n} v_i$	$\boxed{m/T}$
Space-mean speed, v(**A**)	$\boxed{\frac{1}{n}\sum_{i=1}^{n} v_i}$	$[\frac{1}{m}\sum_{j=1}^{m} p_j]^{-1} = \left[\frac{1}{m}\sum_{j=1}^{m}\frac{1}{u_j}\right]^{-1}$
Average pace, p(**A**)	$[\frac{1}{n}\sum_{i=1}^{n} v_i]^{-1}$	$\frac{1}{m}\sum_{j=1}^{m} p_j$
t(**A**)	ndt	$dx\sum_{j=1}^{m} p_j$
d(**A**)	$dt\sum_{i=1}^{n} v_i$	mdx



- Notation
  - v_i = velocity (mi/hr) of vehicle i in section L at time t_0
  - p_j = pace (hr/mi) of vehicle j during duration T at location x_0
  - u_j = velocity (mi/hr) of vehicle j, $u_j = \frac{1}{p_j}$
  - $t(\boldsymbol{A})$ = total time spent in $\boldsymbol{A}$ by all vehicles
  - $d(\boldsymbol{A})$ = total distance traveled by vehicles in $\boldsymbol{A}$
- If traffic is stationary, then the two columns coincide, i.e. same values.
  - Proof: $v = v_i = \frac{1}{p_j}, \forall i, j$
- Then, for instance, density can be measured by counts at a fixed location.

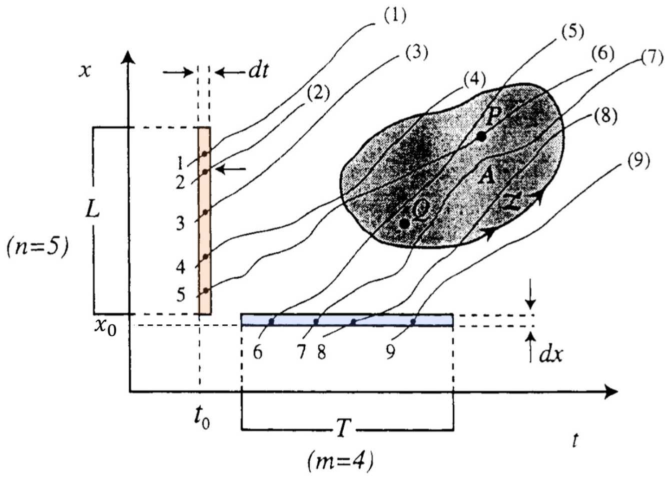
C. Daganzo, *Fundamentals of transportation and traffic operations*, vol. 30. Pergamon Oxford, 1997. Chapter 4: Traffic flow theory.

Outline

1. **Basic assumptions of traffic flow theory**
   a. Key variables
   b. **Time vs space means**
2. Fundamental diagrams (FDs)
3. Highway delay problem

Time and space means

- Space-mean: averages taken at an instant over a space interval
- Time-mean: averages taken at a specific locatio (with time-varying over an interval)
- Speed:
  - $\overline{v}_s$: space-mean speed
  - $\overline{v}_t$: time-mean speed



- Other *vehicle characteristics* can be averaged across space or time. E.g., occupancies (number of persons per vehicle), energy consumption, emissions, etc.
  - There is no a priori reason to expect averages taken across space or time to be the same.
  - Example: You own two cars, they are both driven an equal distance of 100 miles. One gets 20 miles per gallon (mpg), the other 50 mpg. Is the average mpg 35 (i.e. $\frac{50+20}{2}$)?

Time and space means

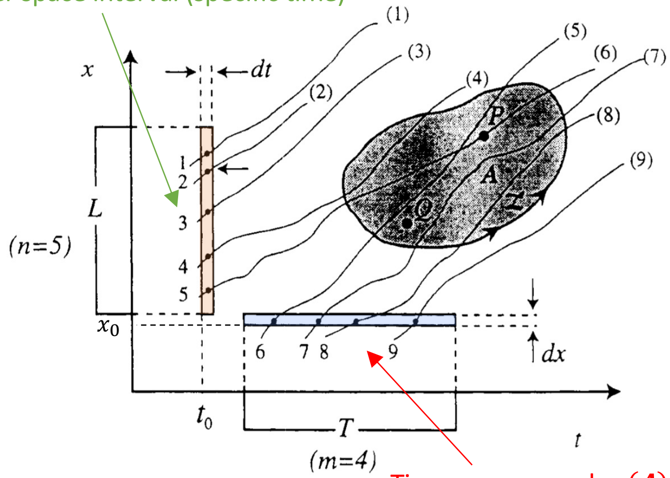
Table 4.1.. Generalized formulas for various traffic characteristics using two observation methods. Boxed expressions correspond to the original definitions introduced in Chapter 1:

	Method of Observation	
	Instantaneous photograph at time t_0 **(section length, L)**	**Observation from a fixed location** x_0 **(duration, T)**
Space-mean speed, v(A)	$\boxed{\frac{1}{n}\sum_{i=1}^{n} v_i}$	$[\frac{1}{m}\sum_{j=1}^{m} p_j]^{-1} = \left[\frac{1}{m}\sum_{j=1}^{m}\frac{1}{u_j}\right]^{-1}$
Time-mean speed $v_t(\boldsymbol{A})$	$\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{v_i}\right]^{-1}$	$\frac{1}{m}\sum_{j=1}^{m}\frac{1}{p_j} = \frac{1}{m}\sum_{j=1}^{m} u_j$

Harmonic mean

Arithmetic mean

Space-mean speed $v(\boldsymbol{A}) \equiv v_s(\boldsymbol{A})$
mean over space interval (specific time)



Time-mean speed $v_t(\boldsymbol{A})$
mean over time duration (specific location)

- Notation:
  - v_i = velocity (mi/hr) of vehicle i
  - p_j = pace (hr/mi) of vehicle j
  - u_j = velocity (mi/hr) of vehicle j, $u_j = \frac{1}{p_j}$
- If traffic is stationary, then time-mean speed = space-mean speed.
  - Proof: $v = v_i = \frac{1}{p_j}, \forall i, j$

C. Daganzo, *Fundamentals of transportation and traffic operations*, vol. 30. Pergamon Oxford, 1997. Chapter 4: Traffic flow theory.

Time and space means in practice

- Time-mean speeds: Often how dual inductance loop detectors in traffic management systems are configured
  - Ex. arithmetic average of vehicle speeds over 20-second intervals
- Space-mean speeds: In nearly all cases of traffic analysis, space-mean speeds should be used
  - Statistically more stable in short segments/durations
  - Weighs slower vehicles' speeds more heavily

Space-mean speed $v(\boldsymbol{A}) \equiv v_s(\boldsymbol{A})$
mean over space interval (specific time)

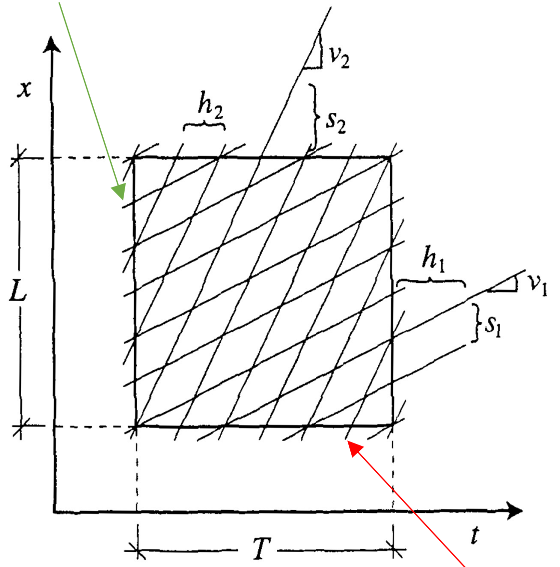


Figure 1.4 Time-space trajectories of two vehicle families.

Time-mean speed $v_t(\boldsymbol{A})$
mean over time duration (specific location)

Turner, Shawn M., William L. Eisele, Robert J. Benz, and Douglas J. Holdener. *Travel time data collection handbook*. No. FHWA-PL-98-035. United States. Federal Highway Administration, 1998. Chapter 1: Introduction. Link.
C. Daganzo, *Fundamentals of transportation and traffic operations*, vol. 30. Pergamon Oxford, 1997. Chapter 1: Time-space diagrams.

Time and space means in practice

- In practice, time-mean and space-mean speeds differ by 1-5%

- Differences are greater when there is more variability in speed (more congestion)

Table 1-1. Comparison of Time-Mean and Space-Mean Speeds

Data Items	Run 1	Run 2	Run 3	Run 4	Run 5	Sum	Average	Variance
Travel Time (sec)	153	103	166	137	127	686	137.2	
Running Time (sec)	142	103	141	137	127	650	130.0	
Stopped Delay Time (sec)	11	0	25	0	0	36	7.2	
Average Travel Speed (km/h)	44.7	66.4	41.2	49.9	53.9	256	51.2	95
Average Running Speed (km/h)	48.1	66.4	48.4	49.9	53.9	n.a.	52.6	

Section Length = 1.9 km

Difference between Time-Mean Speed and Space-Mean Speed
Time-Mean Speed = ∑(speeds) / no. of runs = 256 / 5 = 51.2 km/h
Space-Mean Speed = no. of runs × distance / ∑ (travel times) = 5 x 1.9 / 686 = 49.8 km/h
Therefore, difference = 1.4 km/h

Check Equation 1-5: Time-Mean Speed ≈ 49.8 + 95 / 49.8 ≈ 51.7 km/h ≈ 51.2 km/h

Turner, Shawn M., William L. Eisele, Robert J. Benz, and Douglas J. Holdener. *Travel time data collection handbook*. No. FHWA-PL-98-035. United States. Federal Highway Administration, 1998. Chapter 1: Introduction. Link.